Graded reflection equation algebras and integrable Kondo impurities in the one-dimensional t-J model

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Integrable Kondo impurities in two cases of the one-dimensional $t-J$ model are studied by means of the boundary $\mathbf{Z}_2$-graded quantum inverse scattering method. The boundary K matrices depending on the local magnetic moments of the impurities are presented as nontrivial realizations of the reflection equation algebras in an impurity Hilbert space. Furthermore, these models are solved by using the algebraic Bethe ansatz method and the Bethe ansatz equations are obtained.

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I. INTRODUCTION

The Kondo problem describing the effect due to the exchange interaction between the magnetic impurity and the conduction electrons plays a very important role in condensed matter physics [1]. Wilson [2] developed a very powerful numerical renormalization group approach, and the model was also solved by the coordinate Bethe ansatz method [3,4] which gives the specific heat and magnetization. More recently, a conformal field theory approach was developed by Affleck and Ludwig [5] based on a work by Nozières [6]. In the conventional Kondo problem, the interaction between conduction electrons is discarded, due to the fact that the interacting electron system can be described as a Fermi liquid. Recently there has been substantial research devoted to the investigation of the theory of impurities coupled to Luttinger liquids. Such a problem was first considered by Lee and Toner [7]. By using the perturbative renormalization group theory they found that the Kondo temperature crosses from a generic power law dependence on the Kondo coupling constant to an exponential one in the infinite limit. Afterwards, a "poor man's" scaling procedure was carried out by Furusaki and Nagaosa [8], who found a stable strong coupling fixed point for both antiferromagnetic and ferromagnetic cases. On the other hand, boundary conformal field theory predicts two types of critical behaviours, i.e., either a local Fermi liquid with standard low-temperature thermodynamics or the non-Fermi liquid observed by Furusaki and Nagaosa [8]. However, in order to get a full picture about the critical behaviour of Kondo impurities coupled to Luttinger liquids, some simple integrable models which allow exact solutions are desirable.

Several integrable magnetic or nonmagnetic impurity problems describing impurities embedded in systems of correlated electrons have so far appeared in the literature. Among them are versions of the supersymmetric $t-J$ model with impurities [9–13]. Such an idea to incorporate an impurity into a closed chain dates back to Andrei and Johanesson [14] (see also [15,16]). However, the model thus constructed suffers from the lack of backward scattering and results in a very complicated Hamiltonian which is difficult to be justified on physical grounds. Therefore, as observed by Kane and Fisher [17], it is advantageous to adopt open boundary conditions with the impurities situated at the ends of the chain when studying Kondo impurities coupled to integrable strongly correlated electron systems [18–20].

In this paper, integrable Kondo impurities with arbitrary spin coupled to the one-dimensional $t-J$ open chain are constructed following a formalism recently advocated in [20]. Our new input is to search for integrable boundary K matrices depending on the local magnetic moments of impurities, which arise as a nontrivial realization of the $\mathbf{Z}_2$-graded reflection equation (RE) algebras in a finite dimensional quantum space,which may be interpretated as an impurity Hilbert space. It should be emphasized that our new non-c-number boundary K matrices are highly nontrivial, in the sense that they can not be factorized into the product of a c-number boundary K matrix and the corresponding local monodromy matrices. The models we present are solved by means of the algebraic Bethe ansatz method and the Bethe ansatz equations are derived.

The layout of this paper is the following. We begin by reviewing the $\mathbf{Z}_2$-graded boundary Quantum Inverse Scattering Method (QISM) as formulated in [21]. We then introduce two integrable cases of the one-dimensional $t-J$ model with Kondo impurities on the boundaries. Integrability of the models is established by relating the Hamiltonians to one parameter families of commuting transfer matrices. This is achieved through solving the reflection equations

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for non-c-number solutions. Finally we solve the model by means of the algebraic Bethe ansatz method and derive the Bethe ansatz equations.

II. GRADED REFLECTION EQUATION ALGEBRA AND TRANSFER MATRIX

In this section, we give a brief review about the $\mathbf{Z}_2$-graded boundary quantum inverse scattering method. To begin, let V be a finite-dimensional linear superspace. Let $R \in End(V \otimes V)$ be a solution to the $\mathbf{Z}_2$-graded quantum Yang-Baxter equation (QYBE)

$$R_{12}(u_1 - u_2)R_{13}(u_1 - u_3)R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3)R_{13}(u_1 - u_3)R_{12}(u_1 - u_2). \tag{II.1}$$

Here $R_{jk}(u)$ denotes the matrix on $V \otimes V \otimes V$ acting on the j-th and k-th superspaces and as an identity on the remaining superspace. The variables $u_1,\ u_2$ and u_3 are spectral parameters. The tensor product should be understood in the graded sense, that is the multiplication rule for any homogeneous elements $x,\ y,\ x',\ y' \in EndV$ is given by

$$(x \otimes y)(x' \otimes y') = (-1)^{[y][x']}\,(xx' \otimes yy'), \tag{II.2}$$

where $[x]$ stands for the $\mathbf{Z}_2$-grading of the element x. Let P be the $\mathbf{Z}_2$-graded permutation operator in $V \otimes V$. Then $P(x \otimes y) = (-1)^{[x][y]} y \otimes x,\ \forall x, y \in V$ and $R_{21}(u) = P_{12}R_{12}(u)P_{12}$.

We form the monodromy matrix $T(u)$ for a L-site lattice chain by

$$T(u) = R_{0L}(u) \cdots R_{01}(u). \tag{II.3}$$

Indeed, one may show that $T(u)$ generates a representation of the $\mathbf{Z}_2$-graded quantum Yang-Baxter algebra,

$$R_{12}(u_1 - u_2) \overset{1}{T}(u_1) \overset{2}{T}(u_2) = \overset{2}{T}(u_2) \overset{1}{T}(u_1) R_{12}(u_1 - u_2), \tag{II.4}$$

where for notational convenience we have

$$\overset{1}{T}(u) = T_{13}(u), \qquad \overset{2}{T}(u) = T_{23}(u),$$

and the subscript 3 now labels the quantum superspace $W = V^{\otimes L}$.

In order to describe integrable Kondo impurities in strongly correlated electronic models with open boundary conditions, we need to introduce an appropriate $\mathbf{Z}_2$-graded reflection equation (RE) algebra. We introduce the associative superalgebras $\mathcal{T}_-$ and $\mathcal{T}_+$ defined by the R-matrix and the relations

$$\begin{aligned}
&R_{12}(u_1 - u_2) \overset{1}{\mathcal{T}}_-(u_1) R_{21}(u_1 + u_2) \overset{2}{\mathcal{T}}_-(u_2) = \overset{2}{\mathcal{T}}_-(u_2) R_{12}(u_1 + u_2) \overset{1}{\mathcal{T}}_-(u_1) R_{21}(u_1 - u_2),\\
&R_{21}^{st_1\ ist_2}(-u_1 + u_2) \overset{1}{\mathcal{T}}{}_+^{st_1}(u_1) R_{12}(-u_1 - u_2 + \eta) \overset{2}{\mathcal{T}}{}_+^{ist_2}(u_2)\\
&\qquad\qquad = \overset{2}{\mathcal{T}}{}_+^{ist_2}(u_2) R_{21}(-u_1 - u_2 + \eta) \overset{1}{\mathcal{T}}{}_+^{st_1}(u_1) R_{12}^{st_1\ ist_2}(u_1 - u_2).
\end{aligned} \tag{II.5}$$

Here η is the so-called crossing parameter and st_i stands for the supertransposition taken in the i-th space, whereas ist_i is the inverse operation of st_i. One of the important steps towards correctly formulating the $\mathbf{Z}_2$-graded case is to introduce in the second RE in (II.10) the inverse operation of the supertransposition. In our cases, the R-matrices enjoy the unitarity property,

$$R_{12}(u_1 - u_2)R_{21}(-u_1 + u_2) = 1, \tag{II.6}$$

and the crossing unitarity

$$R_{12}^{st_1}(u_1 - u_2)R_{21}^{st_1}(-u_1 + u_2 + \eta) = 1. \tag{II.7}$$

One can obtain a class of realizations of the superalgebras $\mathcal{T}_+$ and $\mathcal{T}_-$ by choosing $\mathcal{T}_\pm(u)$ to be the form

$$\mathcal{T}_-(u) = T_-(u)\tilde{\mathcal{T}}_-(u)T_-^{-1}(-u), \qquad \mathcal{T}_+^{st}(u) = T_+^{st}(u)\tilde{\mathcal{T}}_+^{st}(u)\left(T_+^{-1}(-u)\right)^{st}, \tag{II.8}$$

with

$$T_-(u) = R_{0M}(u)\cdots R_{01}(u), \quad T_+(u) = R_{0L}(u)\cdots R_{0,M+1}(u), \quad \tilde{\mathcal{T}}_\pm(u) = K_\pm(u), \tag{II.9}$$

where M is any index between 1 and L, and $K_\pm(u)$, called boundary K-matrices, are representations of $\mathcal{T}_\pm$. In the following, without loss of generality, we shall choose $M = L$ so that $\mathcal{T}_+(u) \equiv K_+(u)$.

The K-matrices $K_\pm(u)$ satisfy the same relations as $\mathcal{T}_\pm(u)$, respectively. That is the K-matrices obey the following REs

$$\begin{aligned}
&R_{12}(u_1-u_2)\stackrel{1}{K}_-(u_1)R_{21}(u_1+u_2)\stackrel{2}{K}_-(u_2) = \stackrel{2}{K}_-(u_2)R_{12}(u_1+u_2)\stackrel{1}{K}_-(u_1)R_{21}(u_1-u_2),\\
&R_{21}^{st_1\ ist_2}(-u_1+u_2)\stackrel{1}{K}{}_+^{st_1}(u_1)R_{12}(-u_1-u_2+\eta)\stackrel{2}{K}{}_+^{ist_2}(u_2)\\
&\qquad = \stackrel{2}{K}{}_+^{ist_2}(u_2)R_{21}(-u_1-u_2+\eta)\stackrel{1}{K}{}_+^{st_1}(u_1)R_{12}^{st_1\ ist_2}(-u_1+u_2),
\end{aligned} \tag{II.10}$$

Following Sklyanin's approach [30], one defines the boundary transfer matrix $\tau(u)$ as

$$\tau(u) = str(K_+(u)\mathcal{T}_-(u)) = str\left(K_+(u)T(u)K_-(u)T^{-1}(-u)\right). \tag{II.11}$$

Then it can be shown that [21]

$$[\tau(u_1), \tau(u_2)] = 0. \tag{II.12}$$

Although many attempts have been made to find c-number boundary K matrices, which may be referred to as the fundamental representation, it is no doubt very interesting to search for non-c-number K matrices arising as representations in some Hilbert spaces, which may be interpreted as impurity spaces.

III. INTEGRABLE NON-C-NUMBER BOUNDARY K-MATRICES AND KONDO IMPURITIES IN THE ONE-DIMENSIONAL $T-J$ MODEL

Let $c_{j,\sigma}^\dagger$ and $c_{j,\sigma}$ denote creation and annihilation operators for conduction electrons with spin σ at site j, satisfying the anti-commutation relations given by $\{c_{i,\sigma}^\dagger, c_{j,\sigma'}\} = \delta_{ij}\delta_{\sigma\sigma'}$, where $i, j = 1, 2, \cdots, L$ and $\sigma, \sigma' = \uparrow, \downarrow$. We consider the following type of Hamiltonians describing two magnetic impurities coupled to open t-J chains

$$\begin{aligned}
H = &-t\sum_{j=1,\sigma}^{L-1}\mathcal{P}(c_{j\sigma}^\dagger c_{j+1\sigma} + H.c.)\mathcal{P} + \sum_{j=1}^{L-1}(J\mathbf{S}_j\cdot\mathbf{S}_{j+1} + Vn_jn_{j+1}) +\\
&J_a\mathbf{S}_1\cdot\mathbf{S}_a + V_an_1 + J_b\mathbf{S}_L\cdot\mathbf{S}_b + V_bn_L.
\end{aligned} \tag{III.1}$$

Above, the projector $\mathcal{P} = \prod_{j=1}^L(1-n_{j\uparrow}n_{j\downarrow})$ ensures that double electron occupancies of sites are forbidden; $J_\alpha, V_\alpha(\alpha = a, b)$ are the Kondo coupling constants and the impurity scalar potentials respectively; $\mathbf{S_j}$ as usual is the vector spin operator for the conduction electrons at site j; $\mathbf{S}_\alpha(\alpha = a, b)$ are the local moments with spin-s located at the left and right ends of the system respectively; $n_{j\sigma}$ is the number density operator $n_{j\sigma} = c_{j\sigma}^\dagger c_{j\sigma}$, $n_j = n_{j\uparrow} + n_{j\downarrow}$.

For the choices

$$t = 1, \quad J = 2, \quad V = -\frac{1}{2}, \tag{III.2}$$

it has been shown in refs. [22–24] that the bulk Hamiltonian acquires an underlying supersymmetry algebra given by $gl(2|1)$ in the minimal representation. Throughout we will refer to this case as the supersymmetric t-J model. Integrability of this model on a closed chain with periodic boundary conditions was established independently in works by by Essler and Korepin [25] and Foerster and Karowski [26] by showing that the model can be constructed using the QISM. Furthermore, open chain integrability with appropriate boundary conditions was shown in refs. [27–29]. It is quite interesting to note that although the introduction of integrable impurities we propose below spoils the supersymmetry, there still remains $su(2)$ symmetry in the Hamiltonian (III.1) which maintains conservation of total spin and electron number. We will establish the quantum integrability of the Hamiltonian (III.1) for the special choice of the model parameters (III.2) and

$$J_\alpha = -\frac{2}{(c_\alpha - s)(c_\alpha + s + 1)}, V_\alpha = -\frac{c_\alpha^2 - s(s+1)}{(c_\alpha - s)(c_\alpha + s + 1)}. \tag{III.3}$$

This is achieved by showing that this model can be derived from the QISM. Our result is consistent with the applicability of the coordinate Bethe ansatz method discussed in [18].

Another choice of couplings which leads to an integrable model on the closed periodic chain is given by

$$t = 1, \quad J = -2, \quad V = -\frac{3}{2}, \tag{III.4}$$

as shown by Schlottmann [22] corresponding to an $su(3)$ invariant solution of the Yang-Baxter equation. In this case we can introduce integrable Kondo impurities on the boundary by choosing

$$J_\alpha = \frac{2}{(c_\alpha + s)(c_\alpha - s - 1)}, V_\alpha = \frac{c_\alpha^2 - 1 - s(s+1)}{(c_\alpha + s)(c_\alpha - s - 1)}. \tag{III.5}$$

Below, we describe how the aforementioned integrable cases are obtained.

Let us recall that the local Hamiltonian of the supersymmetric $t - J$ model is derived from an R-matrix satisfying the Yang-Baxter equation which has the form [25,26]

$$R(u) = uI + P \equiv \begin{pmatrix} u-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & u & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & u & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & u-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & u & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u+1 \end{pmatrix}, \tag{III.6}$$

where u is the spectral parameter and

$$P = \sum_{ij} (-1)^{[j]} e_j^i \otimes e_i^j$$

is the form of the $\mathbf{Z}_2$-graded permutation operator in accordance with the rule (II.2). We chose to adopt the $\mathbf{Z}_2$-grading $[1] = [2] = 1$, $[3] = 0$ on the indices labelling the basis vectors.

We now solve (II.10) for $K_+(u)$ and $K_-(u)$. For the quantum R-matrix (III.6), One may check that the matrix $K_-(u)$ given by

$$K_-(u) = \begin{pmatrix} A_-(u) & B_-(u) & 0 \\ C_-(u) & D_-(u) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{III.7}$$

where

$$\begin{aligned} A_-(u) &= \frac{c_a(c_a+1) - u^2 + u - s_a(s_a+1) + 2u\mathbf{S}_a^z}{(c_a + u - s_a)(c_a + u + s_a + 1)}, \\ B_-(u) &= \frac{2u\mathbf{S}_a^-}{(c_a + u - s_a)(c_a + u + s_a + 1)}, \\ C_-(u) &= \frac{2u\mathbf{S}_a^+}{(c_a + u - s_a)(c_a + u + s_a + 1)}, \\ D_-(u) &= \frac{c_a(c_a+1) - u^2 + u - s_a(s_a+1) - 2u\mathbf{S}_a^z}{(c_a + u - s_a)(c_a + u + s_a + 1)}, \end{aligned} \tag{III.8}$$

satisfies the first equation of (II.10)(for the details, see the Appendix). Here $\mathbf{S}^\pm = \mathbf{S}^x \pm i\mathbf{S}^y$. The matrix $K_+(u)$ can be obtained from the isomorphism of the superalgebras $\mathcal{T}_-$ and $\mathcal{T}_+$. Indeed, given a solution K_- of the first equation of (II.10), then $K_+(u)$ defined by

$$K_+^{st}(u) = K_-(-u + \frac{1}{2}) \tag{III.9}$$

is a solution of the second equation of (II.10). The proof follows from some algebraic computations upon substituting (III.9) into the second equation of (II.10) and making use of the properties (II.6) and (II.7) of the R-matrix with $\eta = 1$. Therefore, one may choose the boundary matrix $K_+(u)$ as

$$K_+(u) = \begin{pmatrix} A_+(u) & B_+(u) & 0 \\ C_+(u) & D_+(u) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{III.10}$$

where

$$\begin{aligned} A_+(u) &= \frac{c_b^2 - u^2 - s_b(s_b+1) + (2u-1)\mathbf{S}_b^z}{(c_b+u+s_b)(c_b+u-s_b-1)}, \\ B_+(u) &= \frac{(2u-1)\mathbf{S}_b^-}{(c_b+u+s_b)(c_b+u-s_b-1)}, \\ C_+(u) &= \frac{(2u-1)\mathbf{S}_b^+}{(c_b+u+s_b)(c_b+u-s_b-1)}, \\ D_+(u) &= \frac{c_b^2 - u^2 - s_b(s_b+1) - (2u-1)\mathbf{S}_b^z}{(c_b+u+s_b)(c_b+u-s_b-1)}. \end{aligned} \tag{III.11}$$

It can be shown that the Hamiltonian (III.1) is related to the logarithmic derivative of the transfer matrix $\tau(u)$ with respect to the spectral parameter u at $u = 0$ (up to an additive chemical potential term)

$$-H = \sum_{j=1}^{L-1} h_{j,j+1} + \frac{1}{2}\overset{1}{K'}_-(0) + \frac{str_0 K_+(0) H_{L0}}{str_0 K_+(0)}, \tag{III.12}$$

with

$$h = \frac{d}{du} PR(u).$$

For this case we obtain (III.1) subject to the constraints (III.2, III.3). This implies that this model admits an infinite number of conserved currents thus assuring integrability.

The second choice of integrable couplings results from use of an R-matrix obtained by imposing $\mathbf{Z}_2$-grading to the fundamental $su(3)$ R-matrix and which reads

$$R(u) = \begin{pmatrix} -u-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -u & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -u & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -u-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & u & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u+1 \end{pmatrix}, \tag{III.13}$$

where again u is the spectral parameter and we adopt the same choice for the $\mathbf{Z}_2$-grading of the basis states as before.

We now solve (II.10) for $K_+(u)$ and $K_-(u)$ for this R-matrix (III.13). One may check that the matrix $K_-(u)$ given by

$$K_-(u) = \begin{pmatrix} A_-(u) & B_-(u) & 0 \\ C_-(u) & D_-(u) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{III.14}$$

where

$$\begin{aligned} A_-(u) &= \frac{c_a(c_a-1) - u^2 - u - s_a(s_a+1) - 2u\mathbf{S}_a^z}{(c_a-u+s_a)(c_a-u-s_a-1)}, \\ B_-(u) &= -\frac{2u\mathbf{S}_a^-}{(c_a-u+s_a)(c_a-u-s_a-1)}, \end{aligned}$$

$$C_-(u) = -\frac{2u\mathbf{S}_a^+}{(c_a - u + s_a)(c_a - u - s_a - 1)},$$
$$D_-(u) = \frac{c_a(c_a-1) - u^2 - u - s_a(s_a+1) + 2u\mathbf{S}_a^z}{(c_a - u + s_a)(c_a - u - s_a - 1)}, \quad \text{(III.15)}$$

satisfies the first equation of (II.10). For this case $K_+(u)$ defined by

$$K_+^{st}(u) = K_-(-u - \frac{3}{2}) \quad \text{(III.16)}$$

is a solution of the second equation of (II.10), since the crossing parameter $\eta = -3$. Therefore, we may choose the boundary matrix $K_+(u)$ as

$$K_+(u) = \begin{pmatrix} A_+(u) & B_+(u) & 0 \\ C_+(u) & D_+(u) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{(III.17)}$$

where

$$A_+(u) = -\frac{c_b^2 - u^2 - 2u - s_b(s_b+1) - 1 - (2u+3)\mathbf{S}_b^z}{(c_b - u + s_b - 1)(c_b - u - s_b - 2)},$$
$$B_+(u) = \frac{(2u+3)\mathbf{S}_b^-}{(c_b - u + s_b - 1)(c_b - u - s_b - 2)},$$
$$C_+(u) = \frac{(2u+3)\mathbf{S}_b^+}{(c_b - u + s_b - 1)(c_b - u - s_b - 2)},$$
$$D_+(u) = -\frac{c_b^2 - u^2 - 2u - s_b(s_b+1) - 1 + (2u+3)\mathbf{S}_b^z}{(c_b - u + s_b - 1)(c_b - u - s_b - 2)}. \quad \text{(III.18)}$$

As before, the Hamiltonian in this case is defined as the logarithmic derivative of the transfer matrix at zero spectral parameter giving the same general form (III.1) subject to the constraints (III.4, III.5).

Before concluding this subsection,we would like to point out that the quantum integrability of other two cases corresponding to the same t but with negation of all J's and V's follows from the transformation $c_j^\dagger \to (-1)^j c_j^\dagger$, $c_j \to (-1)^j c_j$.

IV. THE BETHE ANSATZ SOLUTIONS

Having established the quantum integrability of the models, let us first diagonalize the Hamiltonian (III.1) by means of the algebraic Bethe ansatz method [30,31] for the choice of couplings (III.2, III.3). We introduce the 'doubled' monodromy matrix $U(u)$,

$$U(u) = T(u)K_-(u)\tilde{T}(u) \equiv \begin{pmatrix} \mathcal{A}_{11}(u) & \mathcal{A}_{12}(u) & \mathcal{B}_1(u) \\ \mathcal{A}_{21}(u) & \mathcal{A}_{22}(u) & \mathcal{B}_2(u) \\ \mathcal{C}_1(u) & \mathcal{C}_2(u) & \mathcal{D}(u) \end{pmatrix}, \quad \text{(IV.1)}$$

where $\tilde{T}(u) = T^{-1}(-u)$. Substituting into the graded reflection equation (II.10) , we may draw the following commutation relations,

$$\begin{aligned}\check{\mathcal{A}}_{bd}(u_1)\mathcal{C}_c(u_2) = &\frac{(u_1-u_2-1)(u_1+u_2)}{(u_1-u_2)(u_1+u_2+1)} r(u_1+u_2+1)^{eb}_{gh} r(u_1-u_2)^{ih}_{cd} \mathcal{C}_e(u_2)\check{\mathcal{A}}_{gi}(u_1) - \\ &\frac{4u_1u_2}{(u_1+u_2+1)(2u_1+1)(2u_2+1)} r(2u_1+1)^{gb}_{cd}\mathcal{C}_g(u_1)\mathcal{D}(u_2) + \\ &\frac{2u_1}{(u_1-u_2)(2u_1+1)} r(2u_1+1)^{gb}_{id}\mathcal{C}_g(u_1)\check{\mathcal{A}}_{ic}(u_2),\end{aligned} \quad \text{(IV.2)}$$

$$\begin{aligned}\mathcal{D}(u_1)\mathcal{C}_b(u_2) = &\frac{(u_1-u_2-1)(u_1+u_2)}{(u_1-u_2)(u_1+u_2+1)}\mathcal{C}_b(u_2)\mathcal{D}(u_1) + \frac{2u_2}{(u_1-u_2)(2u_2+1)}\mathcal{C}_b(u_1)\mathcal{D}(u_2) \\ &-\frac{1}{u_1+u_2+1}\mathcal{C}_d(u_1)\check{\mathcal{A}}_{db}(u_2).\end{aligned} \quad \text{(IV.3)}$$

Here $\mathcal{A}_{bd}(u) = \check{\mathcal{A}}_{bd}(u) + \frac{1}{2u+1}\delta_{bd}\mathcal{D}(u)$ and the matrix $r(u)$, which in turn satisfies the quantum Yang-Baxter equation, takes the form,

$$r^{bb}_{bb}(u) = 1, \qquad r^{bd}_{bd} = -\frac{1}{u-1}, \qquad r^{bd}_{db}(u) = \frac{u}{u-1}, (b \neq d, b, d = 1, 2). \tag{IV.4}$$

Choosing the Bethe state $|\Omega\rangle$ as

$$|\Omega\rangle = \mathcal{C}_{i_1}(u_1) \cdots \mathcal{C}_{i_N}(u_N)|\Psi\rangle F^{i_1 \cdots i_N}, \tag{IV.5}$$

with $|\Psi\rangle$ being the pseudovacuum, and applying the transfer matrix $\tau(u)$ to the state $|\Omega\rangle$, we have $\tau(u)|\Omega\rangle = \Lambda(u)|\Omega\rangle$, with the eigenvalue,

$$\begin{aligned}\Lambda(u) = {} & \frac{2u-1}{2u+1}\frac{(c_b+u-s_b)}{(c_b+u+s_b)}\frac{(c_b+u+s_b+1)}{(c_b+u-s_b-1)}(-\frac{u+1}{u-1})^L \prod_{j=1}^{N} \frac{(u+u_j)(u-u_j-1)}{(u-u_j)(u+u_j+1)} \\ & -\frac{2u}{2u+1}(-\frac{u^2}{u^2-1})^L \prod_{j=1}^{N} \frac{(u+u_j)(u-u_j-1)}{(u-u_j)(u+u_j+1)} \Lambda^{(1)}(u; \{u_i\}),\end{aligned} \tag{IV.6}$$

provided the parameters $\{u_j\}$ satisfy

$$\frac{2u_j-1}{2u_j}\frac{(c_b+u_j-s_b)}{(c_b+u_j+s_b)}\frac{(c_b+u_j+s_b+1)}{(c_b+u_j-s_b-1)}(\frac{u_j+1}{u_j})^{2L} = \Lambda^{(1)}(u_j; \{u_i\}). \tag{IV.7}$$

Here $\Lambda^{(1)}(u; \{u_i\})$ is the eigenvalue of the transfer matrix $\tau^{(1)}(u)$ for the reduced problem, which arises out of the $r(u)$ matrices from the first term in the right hand side of (IV.2), with the reduced boundary K matrices $K^{(1)}_{\pm}(u)$ as,

$$K^{(1)}_{-}(u) = \begin{pmatrix} A^{(1)}_{-}(u) & B^{(1)}_{-}(u) \\ C^{(1)}_{-}(u) & D^{(1)}_{-}(u) \end{pmatrix}, \tag{IV.8}$$

where

$$\begin{aligned} A^{(1)}_{-}(u) &= \frac{c_a^2 - u^2 - s_a(s_a+1) + (2u+1)\mathbf{S}^z_a}{(c_a+u-s_a)(c_a+u+s_a+1)}, \\ B^{(1)}_{-}(u) &= \frac{(2u+1)\mathbf{S}^-_a}{(c_a+u-s_a)(c_a+u+s_a+1)}, \\ C^{(1)}_{-}(u) &= \frac{(2u+1)\mathbf{S}^+_a}{(c_a+u-s_a)(c_a+u+s_a+1)}, \\ D^{(1)}_{-}(u) &= \frac{c_a^2 - u^2 - s_a(s_a+1) - (2u+1)\mathbf{S}^z_a}{(c_a+u-s_a)(c_a+u+s_a+1)}, \end{aligned} \tag{IV.9}$$

and

$$K^{(1)}_{+}(u) = \begin{pmatrix} A^{(1)}_{+}(u) & B^{(1)}_{+}(u) \\ C^{(1)}_{+}(u) & D^{(1)}_{+}(u) \end{pmatrix}, \tag{IV.10}$$

where

$$\begin{aligned} A^{(1)}_{+}(u) &= \frac{c_b^2 - u^2 - s_b(s_b+1) + (2u-1)\mathbf{S}^z_b}{(c_b+u+s_b)(c_b+u-s_b-1)}, \\ B^{(1)}_{+}(u) &= \frac{(2u-1)\mathbf{S}^-_b}{(c_b+u+s_b)(c_b+u-s_b-1)}, \\ C^{(1)}_{+}(u) &= \frac{(2u-1)\mathbf{S}^+_b}{(c_b+u+s_b)(c_b+u-s_b-1)}, \\ D^{(1)}_{+}(u) &= \frac{c_b^2 - u^2 - s_b(s_b+1) - (2u-1)\mathbf{S}^z_b}{(c_b+u+s_b)(c_b+u-s_b-1)}. \end{aligned} \tag{IV.11}$$

Here $K_-^{(1)}(u)$, the boundary K matrices after the first nesting, follows from the relations,

$$\begin{aligned}\check{\mathcal{A}}_{dd}(u)|\Psi\rangle &\equiv \frac{2u}{2u+1}K_{dd}^{(1)}(u)|\Psi\rangle = (K_-(u)_{dd} - \frac{1}{2u+1})(-\frac{u^2}{u^2-1})^L|\Psi\rangle,\\ \check{\mathcal{A}}_{db}(u)|\Psi\rangle &\equiv \frac{2u}{2u+1}K_{db}^{(1)}(u)|\Psi\rangle = K_-(u)_{db}(-\frac{u^2}{u^2-1})^L|\Psi\rangle.\end{aligned} \tag{IV.12}$$

Indeed, applying the monodromy matrix $T(u)$ and its "adjoint" $\tilde{T}(u)$ to the pseudovacuum, we have

$$\begin{aligned}&T_{dd}(u)|\Psi\rangle = u^L|\Psi\rangle, \quad T_{33}(u)|\Psi\rangle = (u+1)^L|\Psi\rangle,\\ &T_{3d}(u)|\Psi\rangle \neq 0, \quad T_{db}(u)|\Psi\rangle = 0, \quad T_{d3}(u)|\Psi\rangle = 0,\\ &\tilde{T}_{dd}(u)|\Psi\rangle = (-\frac{u}{u^2-1})^L|\Psi\rangle, \quad \tilde{T}_{33}(u)|\Psi\rangle = (-\frac{u+1}{u^2-1})^L|\Psi\rangle,\\ &\tilde{T}_{3d}(u)|\Psi\rangle \neq 0, \quad \tilde{T}_{db}(u)|\Psi\rangle = 0, \quad \tilde{T}_{d3}(u)|\Psi\rangle = 0.\end{aligned} \tag{IV.13}$$

Then we have

$$\begin{aligned}\mathcal{D}(u)|\Psi\rangle &= (-\frac{u+1}{u-1})^L|\Psi\rangle,\\ \mathcal{B}_d(u)|\Psi\rangle &= 0, \qquad \mathcal{C}_d(u)|\Psi\rangle \neq 0,\\ \mathcal{A}_{db}(u)|\Psi\rangle &= (-\frac{u^2}{u^2-1})^L K_-(u)_{db}|\Psi\rangle,\\ \mathcal{A}_{dd}(u)|\Psi\rangle &= (-\frac{u^2}{u^2-1})^L(K_-(u)_{dd} - \frac{1}{2u+1})|\Psi\rangle + \frac{1}{2u+1}(-\frac{u+1}{u-1})^L|\Psi\rangle.\end{aligned} \tag{IV.14}$$

In our calculation, use of the following relations has also been made

$$\begin{aligned}(2u+1)T_{13}(u)\tilde{T}_{31}(u) + T_{11}(u)\tilde{T}_{11} + T_{12}(u)\tilde{T}_{21}(u) &= -(2u-1)\tilde{T}_{31}T_{13}(u) + \tilde{T}_{32}T_{23}(u) + \tilde{T}_{33}T_{33}(u),\\ (2u+1)T_{13}(u)\tilde{T}_{32}(u) + T_{11}(u)\tilde{T}_{12} + T_{12}(u)\tilde{T}_{22}(u) &= -2u\tilde{T}_{32}T_{13}(u),\\ T_{21}(u)\tilde{T}_{11} + T_{22}(u)\tilde{T}_{21}(u) + (2u+1)\tilde{T}_{23}T_{31}(u) &= -2u\tilde{T}_{31}(u)T_{23}(u),\\ T_{21}(u)\tilde{T}_{12}(u) + T_{22}(u)\tilde{T}_{22} + (2u+1)T_{23}(u)\tilde{T}_{31}(u) &= \tilde{T}_{31}T_{13}(u) - (2u-1)\tilde{T}_{32}T_{23}(u) + \tilde{T}_{33}T_{33}(u),\end{aligned} \tag{IV.15}$$

which come from a variant of the (graded) Yang-Baxter algebra (II.4) with the R matrix (III.6),

$$\overset{1}{T}(u)R(2u)\overset{2}{\tilde{T}}(u) = \overset{2}{\tilde{T}}(u)R(2u)\overset{1}{T}(u). \tag{IV.16}$$

Implementing the change $u \to u + \frac{1}{2}$ with respect to the original problem, one may check that these boundary K matrices satisfy the reflection equations for the reduced problem. After some algebra, the reduced transfer matrix $\tau^{(1)}(u)$ may be recognized as that for the N-site inhomogeneous XXX spin-$\frac{1}{2}$ open chain with two impurities of arbitrary spin on the boundaries, which may be diagonalized following Ref. [30]. Here we merely give the final result,

$$\begin{aligned}\Lambda^{(1)}(u;\{u_j\}) = \frac{(c_b+u-s_b)}{(c_b+u+s_b)}\frac{(c_b+u+s_b+1)}{(c_b+u-s_b-1)}\prod_{\alpha=a,b}\frac{u-c_\alpha-s_\alpha}{u+c_\alpha+s_\alpha+1}\{\frac{2u-1}{2u}\prod_{m=1}^{M}\frac{(u-v_m+\frac{3}{2})(u+v_m+\frac{1}{2})}{(u-v_m+\frac{1}{2})(u+v_m-\frac{1}{2})}\\ +\frac{2u+1}{2u}\prod_{\alpha=a,b}\frac{(u-c_\alpha+s_\alpha)}{(u-c_\alpha-s_\alpha)}\frac{(u+c_\alpha+s_\alpha)}{(u+c_\alpha-s_\alpha)}\prod_{j=1}^{N}\frac{(u-u_j)(u+u_j+1)}{(u-u_j-1)(u+u_j)}\prod_{m=1}^{M}\frac{(u-v_m-\frac{1}{2})(u+v_m-\frac{3}{2})}{(u-v_m+\frac{1}{2})(u+v_m-\frac{1}{2})}\},\end{aligned} \tag{IV.17}$$

provided the parameters $\{v_m\}$ satisfy

$$\prod_{\alpha=a,b}\frac{(v_m+c_\alpha-s_\alpha-\frac{1}{2})(v_m-c_\alpha-s_\alpha-\frac{1}{2})}{(v_m+c_\alpha+s_\alpha-\frac{1}{2})(v_m-c_\alpha+s_\alpha-\frac{1}{2})}\prod_{j=1}^{N}\frac{(v_m-u_j-\frac{3}{2})(v_m+u_j-\frac{1}{2})}{(v_m-u_j-\frac{1}{2})(v_m+u_j+\frac{1}{2})} = \prod_{\substack{k=1\\k\neq m}}^{M}\frac{(v_m-v_k-1)(v_m+v_k-2)}{(v_m-v_k+1)(v_m+v_k)}. \tag{IV.18}$$

After a shift of the parameters $u_j \to u_j - \frac{1}{2}, v_m \to v_m + \frac{1}{2}$,the Bethe ansatz equations (IV.7) and (IV.18) may be rewritten as follows

$$(\frac{u_j+\frac{1}{2}}{u_j-\frac{1}{2}})^{2L}\prod_{\alpha=a,b}\frac{u_j+c_\alpha+s_\alpha+\frac{1}{2}}{u_j-c_\alpha-s_\alpha-\frac{1}{2}} = \prod_{m=1}^{M}\frac{u_j-v_m+\frac{1}{2}}{u_j-v_m-\frac{1}{2}}\frac{u_j+v_m+\frac{1}{2}}{u_j+v_m-\frac{1}{2}},$$

$$\prod_{\alpha=a,b}\frac{v_m-c_\alpha-s_\alpha}{v_m-c_\alpha+s_\alpha}\frac{v_m+c_\alpha-s_\alpha}{v_m+c_\alpha+s_\alpha}\prod_{j=1}^{N}\frac{(v_m-u_j-\frac{1}{2})}{(v_m-u_j+\frac{1}{2})}\frac{(v_m+u_j-\frac{1}{2})}{(v_m+u_j+\frac{1}{2})} = \prod_{\substack{k=1\\k\neq m}}\frac{(v_m-v_k-1)}{(v_m-v_k+1)}\frac{(v_m+v_k-1)}{(v_m+v_k+1)},$$

with the corresponding energy eigenvalue E of the model

$$E = -\sum_{j=1}^{N}\frac{1}{u_j^2-\frac{1}{4}}. \tag{IV.19}$$

It should be pointed out that when $s_\alpha = \frac{1}{2}$,the above results reduce to those obtained in Ref. [20],which in turn provides an algebraic interpretation for the applicability of the coordinate Bethe ansatz method [18].

We now perform the algebraic Bethe ansatz procedure for the couplings (III.4, III.5). We introduce the 'doubled' monodromy matrix $U(u)$

$$U(u) = T(u)K_-(u)\tilde{T}(u) \equiv \begin{pmatrix} \mathcal{A}_{11}(u) & \mathcal{A}_{12}(u) & \mathcal{B}_1(u) \\ \mathcal{A}_{21}(u) & \mathcal{A}_{22}(u) & \mathcal{B}_2(u) \\ \mathcal{C}_1(u) & \mathcal{C}_2(u) & \mathcal{D}(u) \end{pmatrix}, \tag{IV.20}$$

where $\tilde{T}(u) = T^{-1}(-u)$. Substituting into the reflection equation (II.10), we find the following commutation relations

$$\begin{aligned}\check{\mathcal{A}}_{bd}(u_1)\mathcal{C}_c(u_2) &= \frac{(u_1-u_2+1)(u_1+u_2+2)}{(u_1-u_2)(u_1+u_2+1)}r(u_1+u_2+1)^{eb}_{gh}r(u_1-u_2)^{ih}_{cd}\mathcal{C}_e(u_2)\check{\mathcal{A}}_{gi}(u_1) + \\ &\frac{4(u_1+1)u_2}{(u_1+u_2+1)(2u_1+1)(2u_2+1)}r(2u_1+1)^{gb}_{cd}\mathcal{C}_g(u_1)\mathcal{D}(u_2) - \\ &\frac{2(u_1+1)}{(u_1-u_2)(2u_1+1)}r(2u_1+1)^{gb}_{id}\mathcal{C}_g(u_1)\check{\mathcal{A}}_{ic}(u_2), \qquad \text{(IV.21)}\\ \mathcal{D}(u_1)\mathcal{C}_b(u_2) &= \frac{(u_1-u_2-1)(u_1+u_2)}{(u_1-u_2)(u_1+u_2+1)}\mathcal{C}_b(u_2)\mathcal{D}(u_1) + \frac{2u_2}{(u_1-u_2)(2u_2+1)}\mathcal{C}_b(u_1)\mathcal{D}(u_2) \\ &-\frac{1}{u_1+u_2+1}\mathcal{C}_d(u_1)\check{\mathcal{A}}_{db}(u_2). \qquad \text{(IV.22)}\end{aligned}$$

Here $\mathcal{A}_{bd}(u) = \check{\mathcal{A}}_{bd}(u) + \frac{1}{2u+1}\delta_{bd}\mathcal{D}(u)$ and the matrix $r(u)$, which in turn satisfies the quantum Yang-Baxter equation, takes the form,

$$r^{bb}_{bb}(u) = 1, \qquad r^{bd}_{bd} = \frac{1}{u+1}, \qquad r^{bd}_{db}(u) = \frac{u}{u+1}, (b\neq d, b, d = 1, 2). \tag{IV.23}$$

Choosing the Bethe state $|\Omega\rangle$ as

$$|\Omega\rangle = \mathcal{C}_{i_1}(u_1)\cdots\mathcal{C}_{i_N}(u_N)|\Psi\rangle F^{i_1\cdots i_N}, \tag{IV.24}$$

with $|\Psi\rangle$ being the pseudovacuum, and applying the transfer matrix $\tau(u)$ to the state $|\Omega\rangle$, we have $\tau(u)|\Omega\rangle = \Lambda(u)|\Omega\rangle$, with the eigenvalue

$$\begin{aligned}\Lambda(u) &= \frac{2u+3}{2u+1}\frac{(c_b-u+s_b)}{(c_b-u+s_b-1)}\frac{(c_b-u-s_b-1)}{(c_b-u-s_b-2)}(-\frac{u+1}{u-1})^L\prod_{j=1}^{N}\frac{(u+u_j)(u-u_j-1)}{(u-u_j)(u+u_j+1)} \\ &-\frac{2u}{2u+1}(-\frac{u^2}{u^2-1})^L\prod_{j=1}^{N}\frac{(u+u_j+2)(u-u_j+1)}{(u-u_j)(u+u_j+1)}\Lambda^{(1)}(u;\{u_i\}), \qquad \text{(IV.25)}\end{aligned}$$

provided the parameters $\{u_j\}$ satisfy

$$\frac{2u_j+3}{2u_j+2}\frac{(c_b-u_j+s_b)}{(c_b-u_j+s_b-1)}\frac{(c_b-u_j-s_b-1)}{(c_b-u_j-s_b-2)}(\frac{u_j+1}{u_j})^{2L}\prod_{\substack{i=1\\ i\neq j}}^{M}\frac{(u_j+u_i)(u_j-u_i-1)}{(u_j+u_i+2)(u_j-u_i+1)}=-\Lambda^{(1)}(u_j;\{u_i\}). \quad \text{(IV.26)}$$

Here $\Lambda^{(1)}(u;\{u_i\})$ is the eigenvalue of the transfer matrix $\tau^{(1)}(u)$ for the reduced problem, which arises out of the $r(u)$ matrices from the first term in the right hand side of (IV.21), with the reduced boundary K matrices $K_{\pm}^{(1)}(u)$ as

$$K_-^{(1)}(u)=\begin{pmatrix} A_-^{(1)}(u) & B_-^{(1)}(u)\\ C_-^{(1)}(u) & D_-^{(1)}(u)\end{pmatrix}, \quad \text{(IV.27)}$$

where

$$\begin{aligned}
A_-^{(1)}(u)&=\frac{c_a^2-u^2-2u-s_a(s_a+1)-1-(2u+1)\mathbf{S}_a^z}{(c_a-u+s_a)(c_a-u-s_a-1)},\\
B_-^{(1)}(u)&=-\frac{(2u+1)\mathbf{S}_a^-}{(c_a-u+s_a)(c_a-u-s_a-1)},\\
C_-^{(1)}(u)&=-\frac{(2u+1)\mathbf{S}_a^+}{(c_a-u+s_a)(c_a-u-s_a-1)},\\
D_-^{(1)}(u)&=\frac{c_a^2-u^2-2u-s_a(s_a+1)-1+(2u+1)\mathbf{S}_a^z}{(c_a-u+s_a)(c_a-u-s_a-1)},
\end{aligned} \quad \text{(IV.28)}$$

and

$$K_+^{(1)}(u)=\begin{pmatrix} A_+^{(1)}(u) & B_+^{(1)}(u)\\ C_+^{(1)}(u) & D_+^{(1)}(u)\end{pmatrix}, \quad \text{(IV.29)}$$

where

$$\begin{aligned}
A_+^{(1)}(u)&=-\frac{c_b^2-u^2-2u-s_b(s_b+1)-1-(2u+3)\mathbf{S}_b^z}{(c_b-u+s_b-1)(c_b-u-s_b-2)},\\
B_+^{(1)}(u)&=\frac{(2u+3)\mathbf{S}_b^-}{(c_b-u+s_b-1)(c_b-u-s_b-2)},\\
C_+^{(1)}(u)&=\frac{(2u+3)\mathbf{S}_b^+}{(c_b-u+s_b-1)(c_b-u-s_b-2)},\\
D_+^{(1)}(u)&=-\frac{c_b^2-u^2-2u-s_b(s_b+1)-1+(2u+3)\mathbf{S}_b^z}{(c_b-u+s_b-1)(c_b-u-s_b-2)},
\end{aligned} \quad \text{(IV.30)}$$

Here $K_-^{(1)}(u)$,the boundary K matrices after the first nesting, follows from the relations,

$$\begin{aligned}
\check{\mathcal{A}}_{dd}(u)|\Psi\rangle &\equiv \frac{2u}{2u+1}K_{dd}^{(1)}(u)|\Psi\rangle=(K_-(u)_{dd}-\frac{1}{2u+1})(-\frac{u^2}{u^2-1})^L|\Psi\rangle,\\
\check{\mathcal{A}}_{db}(u)|\Psi\rangle &\equiv \frac{2u}{2u+1}K_{db}^{(1)}(u)|\Psi\rangle=K_-(u)_{db}(-\frac{u^2}{u^2-1})^L|\Psi\rangle
\end{aligned} \quad \text{(IV.31)}$$

Indeed,applying the monodromy matrix $T(u)$ and its "adjoint" $\tilde{T}(u)$ to the pseudovacuum,we have

$$\begin{aligned}
&T_{dd}(u)|\Psi\rangle=u^L|\Psi\rangle, \quad T_{33}(u)|\Psi\rangle=(u+1)^L|\Psi\rangle,\\
&T_{3d}(u)|\Psi\rangle\neq 0, \quad T_{db}(u)|\Psi\rangle=0, \quad T_{d3}(u)|\Psi\rangle=0,\\
&\tilde{T}_{dd}(u)|\Psi\rangle=(-\frac{u}{u^2-1})^L|\Psi\rangle, \quad \tilde{T}_{33}(u)|\Psi\rangle=(-\frac{u+1}{u^2-1})^L|\Psi\rangle,\\
&\tilde{T}_{3d}(u)|\Psi\rangle\neq 0, \quad \tilde{T}_{db}(u)|\Psi\rangle=0, \quad \tilde{T}_{d3}(u)|\Psi\rangle=0,
\end{aligned} \quad \text{(IV.32)}$$

Then we have

$$
\begin{aligned}
\mathcal{D}(u)|\Psi\rangle &= (-\frac{u+1}{u-1})^L|\Psi\rangle,\\
\mathcal{B}_d(u)|\Psi\rangle &= 0, \qquad \mathcal{C}_d(u)|\Psi\rangle \neq 0,\\
\mathcal{A}_{db}(u)|\Psi\rangle &= (-\frac{u^2}{u^2-1})^L K_-(u)_{db}|\Psi\rangle,\\
\mathcal{A}_{dd}(u)|\Psi\rangle &= (-\frac{u^2}{u^2-1})^L (K_-(u)_{dd} - \frac{1}{2u+1})|\Psi\rangle + \frac{1}{2u+1}(-\frac{u+1}{u-1})^L|\Psi\rangle
\end{aligned}
\tag{IV.33}
$$

In our calculation,use have also been made of the following relations,

$$
\begin{aligned}
&(2u+1)T_{13}(u)\tilde{T}_{31}(u) + T_{11}(u)\tilde{T}_{11} + T_{12}(u)\tilde{T}_{21}(u) = (2u+1)\tilde{T}_{31}T_{13}(u) + \tilde{T}_{32}T_{23}(u) + \tilde{T}_{33}T_{33}(u),\\
&(2u+1)T_{13}(u)\tilde{T}_{32}(u) + T_{11}(u)\tilde{T}_{12} + T_{12}(u)\tilde{T}_{22}(u) = 2u\tilde{T}_{32}T_{13}(u),\\
&T_{21}(u)\tilde{T}_{11} + T_{22}(u)\tilde{T}_{21}(u) + (2u+1)\tilde{T}_{23}T_{31}(u) = 2u\tilde{T}_{31}(u)T_{23}(u),\\
&T_{21}(u)\tilde{T}_{12}(u) + T_{22}(u)\tilde{T}_{22} + (2u+1)T_{23}(u)\tilde{T}_{31}(u) = \tilde{T}_{31}T_{13}(u) + (2u+1)\tilde{T}_{32}T_{23}(u) + \tilde{T}_{33}T_{33}(u)
\end{aligned}
\tag{IV.34}
$$

which come from a variant of the (graded) Yang-Baxter algebra (II.4) with the quantum R matrix (III.13),

$$
\overset{1}{T}(u)R(2u)\overset{2}{\tilde{T}}(u) = \overset{2}{\tilde{T}}(u)R(2u)\overset{1}{T}(u).
\tag{IV.35}
$$

Implementing the change $u \to u + \frac{1}{2}$ with respect to the original problem, one may check that these boundary K matrices satisfy the reflection equations for the reduced problem. After some algebra, the reduced transfer matrix $\tau^{(1)}(u)$ may be recognized as that for the N-site inhomogeneous XXX spin-$\frac{1}{2}$ open chain with two impurities of arbitrary spin on the boundaries, which may be diagonalized following Ref. [30]. Here we merely give the final result,

$$
\begin{aligned}
\Lambda^{(1)}(u;\{u_j\}) = &-\frac{(c_b-u+s_b)}{(c_b-u+s_b-1)}\frac{(c_b-u-s_b-1)}{(c_b-u-s_b-2)}\prod_{\alpha=a,b}\frac{u+c_\alpha+s_\alpha+1}{u-c_\alpha-s_\alpha}\\
&\{\frac{2u+3}{2u+2}\prod_{m=1}^{M}\frac{(u-v_m-1)(u+v_m+1)}{(u-v_m)(u+v_m+2)} + \frac{2u+1}{2u+2}\prod_{\alpha=a,b}\frac{(u-c_\alpha-s_\alpha+1)}{(u-c_\alpha+s_\alpha+1)}\frac{(u+c_\alpha-s_\alpha+1)}{(u+c_\alpha+s_\alpha+1)}\\
&\prod_{j=1}^{N}\frac{(u-u_j)(u+u_j+1)}{(u-u_j+1)(u+u_j+2)}\prod_{m=1}^{M}\frac{(u-v_m+1)(u+v_m+3)}{(u-v_m)(u+v_m+2)}\},
\end{aligned}
\tag{IV.36}
$$

provided the parameters $\{v_m\}$ satisfy

$$
\prod_{\alpha=a,b}\frac{(v_m+c_\alpha-s_\alpha+1)(v_m-c_\alpha-s_\alpha+1)}{(v_m+c_\alpha+s_\alpha+1)(v_m-c_\alpha+s_\alpha+1)}\prod_{j=1}^{N}\frac{(v_m-u_j)(v_m+u_j+1)}{(v_m-u_j+1)(v_m+u_j+2)} = \prod_{\substack{k=1\\k\neq m}}^{M}\frac{(v_m-v_k-1)(v_m+v_k+1)}{(v_m-v_k+1)(v_m+v_k+3)}.
\tag{IV.37}
$$

After a shift of the parameters $u_j \to u_j - \frac{1}{2}, v_m \to v_m - 1$,the Bethe ansatz equations (IV.26) and (IV.37) may be rewritten as follows

$$
\begin{aligned}
&(\frac{u_j+\frac{1}{2}}{u_j-\frac{1}{2}})^{2L}\prod_{\alpha=a,b}\frac{u_j-c_\alpha-s_\alpha-\frac{1}{2}}{u_j+c_\alpha+s_\alpha+\frac{1}{2}}\prod_{\substack{i=1\\i\neq j}}^{N}\frac{(u_j-u_i-1)(u_j+u_i-1)}{(u_j-u_i+1)(u_j+u_i+1)} = \prod_{m=1}^{M}\frac{u_j-v_m-\frac{1}{2}}{u_j-v_m+\frac{1}{2}}\frac{u_j+v_m-\frac{1}{2}}{u_j+v_m+\frac{1}{2}},\\
&\prod_{\alpha=a,b}\frac{v_m-c_\alpha-s_\alpha}{v_m-c_\alpha+s_\alpha}\frac{v_m+c_\alpha-s_\alpha}{v_m+c_\alpha+s_\alpha}\prod_{j=1}^{N}\frac{(v_m-u_j-\frac{1}{2})}{(v_m-u_j+\frac{1}{2})}\frac{(v_m+u_j-\frac{1}{2})}{(v_m+u_j+\frac{1}{2})} = \prod_{\substack{k=1\\k\neq m}}\frac{(v_m-v_k-1)}{(v_m-v_k+1)}\frac{(v_m+v_k-1)}{(v_m+v_k+1)},
\end{aligned}
$$

with the corresponding energy eigenvalue E of the model

$$
E = -\sum_{j=1}^{N}\frac{1}{u_j^2-\frac{1}{4}}.
\tag{IV.38}
$$

V. CONCLUSION

In this paper, we have studied an integrable Kondo problem describing two impurities coupled to the one-dimensional $t-J$ open chain for specific couplings. The quantum integrability of the system follows from the fact that the Hamiltonian may be derived from a one-parameter family of commuting transfer matrices. Moreover, the Bethe Ansatz equations are obtained by means of the algebraic Bethe ansatz approach. It should be emphasized that the boundary K matrices found here are highly nontrivial, since they can not be factorized into the product of a c-number K matrix and the local monodromy matrices. However, it is still possible to introduce a "singular" local monodromy matrix $\tilde{L}(u)$ and express the boundary K matrix $K_-(u)$ as

$$K_-(u) = \tilde{L}(u)\tilde{L}^{-1}(-u), \tag{V.1}$$

where

$$\tilde{L}(u) = \begin{pmatrix} u - c_a - 1 - \mathbf{S}^z_\alpha & -\mathbf{S}^-_\alpha & 0 \\ -\mathbf{S}^+_\alpha & u - c_a - 1 + \mathbf{S}^z_\alpha & 0 \\ 0 & 0 & \epsilon \end{pmatrix}, \tag{V.2}$$

which constitutes a realization of the (graded) Yang-Baxter algebra (II.4) when ϵ tends to 0. The implication of such a singular factorization deserves further investigation. Indeed, this implies that integrable Kondo impurities discussed here appear to be, in some sense, related to a singular realization of the (graded) Yang-Baxter algebra, which in turn reflects a hidden six-vertex XXX symmetry in the original quantum R-matrix. Therefore, one may expect that the formalism presented here may be applied to other physically interesting strongly correlated electron systems, such as the supersymmetric extended Hubbard model [32] and the supersymmetric U model [33]. However,our construction is not applicable to the one-dimensional Hubbard model and the one-dimensional Bariev model,although a hidden six-vertex symmetry occurs in these two physically interesting strongly correlated electron systems [34]. Moreover,the singularity of the local monodromy matrix (V.2) implies that we can not apply it to construct a closed $t-J$ chain interacting with integrable Kondo-like impurities. This is different from the conclusion by Zvyagin and Schlottmann [11,35], who claimed that integrable magnetic impurities exist in the closed $t-J$ and Hubbard chains.

As shown in [19], one can put integrable Kondo impurities on the boundaries of the δ-function interaction electron gas. Obviously, there should be no problem in applying our construction to this model. Another question is to extend the present construction to the q-deformed case. This will lead us to integrable anisotropic Kondo impurities coupled to the q-deformed version of the $t-J$ open chain.

In concluding, we would like to point out that it will be interesting to carry out the calculation of thermodynamic equilibrium properties of the model under consideration, based on the Bethe ansatz equations presented here. Especially, it is desirable to calculate the finite-size spectrum analytically, which, together with the predictions of the boundary conformal field thery, will allow us to draw various critical behaviour properties. Also, our construction may shed new light on a long-standing problem about the quantum integrability of the conventional Kondo model by QISM, given it has been solved using the coordinate Bethe ansatz for a long time [3,4].

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APPENDIX A: DERIVATION OF THE NON-C-NUMBER BOUNDARY K-MATRICES

In this appendix, we sketch the procedure of solving the ($\mathbf{Z}_2$-graded) RE for $K_-(u)$. To describe integrable Kondo impurites coupled with the one-dimensional $t-J$ open chain,it is reasonable to assume that

$$K_-(u) = \begin{pmatrix} A(u) & B(u) & 0 \\ C(u) & D(u) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.1}$$

Throughout, we have omitted all the subscrips for brevity, reflecting that the fermionic degrees of freedom do not occur, as it should be for a magnetic impurity. For the R-matrix (III.6), one may get from the RE (II.10) 33 functional equations, of which 11 are identities. After some algebraic analysis, together with the $su(2)$ symmetry, we may assume that

$$
\begin{aligned}
A(u) &= \alpha(u) + \beta(u)\mathbf{S}^z, \quad B(u) = \beta(u)\mathbf{S}^-, \\
C(u) &= \beta(u)\mathbf{S}^+, \quad D(u) = \alpha(u) - \beta(u)\mathbf{S}^z.
\end{aligned} \tag{A.2}
$$

There are two equations automatically satisfied, leaving only 20 equations left to be solved

$$
\begin{aligned}
&A(u_1)B(u_2) + B(u_1)D(u_2) = A(u_2)B(u_1) + B(u_2)D(u_1), \\
&C(u_1)A(u_2) + D(u_1)C(u_2) = C(u_2)A(u_1) + D(u_2)C(u_1), \\
&u_-(A(u_1)B(u_2) + B(u_1)D(u_2)) = u_+(B(u_1) - B(u_2)), \\
&u_-(A(u_2)B(u_1) + B(u_2)D(u_1)) = u_+(B(u_1) - B(u_2)), \\
&u_-(C(u_1)A(u_2) + D(u_1)C(u_2)) = u_+(C(u_1) - C(u_2)), \\
&u_-(C(u_2)A(u_1) + D(u_2)C(u_1)) = u_+(C(u_1) - C(u_2)), \\
&u_-(A(u_1)A(u_2) + B(u_1)C(u_2) - 1) = u_+(A(u_1) - A(u_2)), \\
&u_-(A(u_2)A(u_1) + B(u_2)C(u_1) - 1) = u_+(A(u_1) - A(u_2)), \\
&u_-(C(u_1)B(u_2) + D(u_1)D(u_2) - 1) = u_+(D(u_1) - D(u_2)), \\
&u_-(C(u_2)B(u_1) + D(u_2)D(u_1) - 1) = u_+(D(u_1) - D(u_2)), \\
&u_-((u_+ - 1)B(u_1)D(u_2) - A(u_1)B(u_2)) = u_+((u_- - 1)D(u_2)B(u_1) + D(u_1)B(u_2)), \\
&u_-((u_+ - 1)C(u_1)A(u_2) - D(u_1)C(u_2)) = u_+((u_- - 1)A(u_2)C(u_1) + A(u_1)C(u_2)), \\
&u_-((u_+ - 1)A(u_2)B(u_1) - B(u_2)D(u_1)) = u_+((u_- - 1)B(u_1)A(u_2) + B(u_2)A(u_1)), \\
&u_-((u_+ - 1)D(u_2)C(u_1) - C(u_2)A(u_1)) = u_+((u_- - 1)C(u_1)D(u_2) + C(u_2)D(u_1)), \\
&u_-(A(u_1)A(u_2) - (u_+ - 1)B(u_1)C(u_2) + (u_+ - 1)C(u_2)B(u_1) - D(u_2)D(u_1)) = u_+(D(u_2)A(u_1) - D(u_1)A(u_2)) \\
&u_-(A(u_2)A(u_1) - (u_+ - 1)B(u_2)C(u_1) + (u_+ - 1)C(u_1)B(u_2) - D(u_1)D(u_2)) = u_+(A(u_1)D(u_2) - A(u_2)D(u_1)). \\
&A(u_1)B(u_2) + u_+u_-D(u_1)B(u_2) + (u_- - 1)A(u_2)B(u_1) = (u_+ - 1)(B(u_1)D(u_2) + (u_- - 1)B(u_2)D(u_1)), \\
&D(u_1)C(u_2) + u_+u_-A(u_1)C(u_2) + (u_- - 1)D(u_2)C(u_1) = (u_+ - 1)(C(u_1)A(u_2) + (u_- - 1)C(u_2)A(u_1)), \\
&B(u_2)D(u_1) + u_+u_-B(u_2)A(u_1) + (u_- - 1)B(u_1)D(u_2) = (u_+ - 1)(A(u_2)B(u_1) + (u_- - 1)A(u_1)B(u_2)), \\
&C(u_2)A(u_1) + u_+u_-C(u_2)D(u_1) + (u_- - 1)C(u_1)A(u_2) = (u_+ - 1)(D(u_2)C(u_1) + (u_- - 1)D(u_1)C(u_2)),
\end{aligned}
$$

with $u_+ = u_1 + u_2, u_- = u_1 - u_2$. Substituting (A.2) into these equations, we find that all these equations are reduced to the following three equations

$$
\begin{aligned}
u_+(\alpha(u_1) - \alpha(u_2)) &= u_-(-1 + \alpha(u_1)\alpha(u_2) + s(s+1)\beta(u_1)\beta(u_2)), \\
u_+(\beta(u_1) - \beta(u_2)) &= u_-(\alpha(u_1)\beta(u_2) + \alpha(u_2)\beta(u_1) - \beta(u_1)\beta(u_2)), \\
u_+(\alpha(u_2)\beta(u_1) - \alpha(u_1)\beta(u_2)) &= u_-(\alpha(u_1)\beta(u_2) + \alpha(u_2)\beta(u_1)) + u_-(u_+ - 1)\beta(u_1)\beta(u_2)).
\end{aligned} \tag{A.3}
$$

Taking the limit $u_1 \to u_2$,these equations become

$$
\begin{aligned}
\frac{d\alpha(u)}{du} &= \frac{1}{2u}(-1 + \alpha(u)^2 + s(s+1)\beta(u)^2), \\
\frac{d\beta(u)}{du} &= \frac{1}{2u}(2\alpha(u)\beta(u) - \beta(u)^2), \\
\alpha(u)\frac{d\beta(u)}{du} - \beta(u)\frac{d\alpha(u)}{du} &= \frac{1}{2u}(2\alpha(u)\beta(u) + (2u-1)\beta(u)^2).
\end{aligned} \tag{A.4}
$$

Solving the first two equations, we have

$$
\alpha(u) = \frac{(c_1c_2 - u^2)(2s+1) + (c_2 - c_1)u}{(2s+1)(c_1 - u)(c_2 - u)}, \qquad \beta(u) = \frac{2(c_2 - c_1)u}{(2s+1)(c_1 - u)(c_2 - u)}, \tag{A.5}
$$

where c_1 and c_2 are integration constants. Substituting these results into the third equation in (A.4),we may establish a relation between c_i: $c_2 = c_1 + 2s + 1$. This is nothing but the non-c-number boundary K matrix (III.7) (after a redefinition of the constant: $c_1 \to -c - s - 1$).

A similar construction also works for the quantum R matrix (III.13).

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